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# The vertical gradient of normal gravity and surface geometry

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***Abstract***
*The vertical gradient of normal gravity (VGNG) plays a significant role in several applications and is an interesting quantity from a pure theoretical point of view as well. In this work the VGNG will be studied from a theoretical point of view, therefore the expression of the VGNG at a point P on the Earth's physical surface in geodetic coordinates is presented. Since the value VGNG also depends on the mean curvature of the normal equipotential surfaces, an effort has been made to express the fundamental quantities $E^U$, $F^U$, $G^U$, $L^U$, $M^U$ and $N^U$ at point P in a specific form. These quantities are expressed as combinations of the fundamental quantities of the ellipsoid of revolution, its mean curvature, Gauss curvature, and normal reduction. The fundamental quantities of the ellipsoid of revolution are determined at a point Q which is the projection of a point P, on the ellipsoid along the vertical line.*
*The fundamental quantities of the ellipsoid, its mean curvature and Gauss curvature and the geometric height of the chosen point P represent the geometric part of the quantities $E^U$, $F^U$, $G^U$, $L^U$, $M^U$ and $N^U$. The value of the normal reduction at point P represents the physical part of those quantities (i.e. $E^U$, $F^U$, $G^U$, $L^U$, $M^U$ and $N^U$). The aforementioned fundamental quantities are very complicated functions expressed in geodetic coordinates. Thus we deduce that the significant complexity of the geometry of the normal equipotential surfaces results in an equivalent complexity of VGNG as a function of geodetic coordinates. Finally the effect of the geometry of the ellipsoid of revolution on the VGNG is examined.*

***Keywords:*** *Vertical gradient, normal gravity, ellipsoid, equipotential surfaces, mean curvature, Gauss curvature.*

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## I. Introduction

The vertical gradient of gravity (VGG) is a significant quantity in Geophysics and Geodesy. It shows that the variation of gravity at a chosen point P along the direction of the plumbline depends on physical quantities (Earth's mean angular velocity, and gravity magnitude) and a geometric quantity. The geometric quantity is the mean curvature of the equipotential surface at point P. The expression of the vertical gradient of gravity shows an elegant relation between Geometry and Physics.

The value of the VGG can be estimated practically (Vajda et al, 2015) by relative gravity meters observing on the ground and at a certain height above the benchmark. For example, it is necessary for the determination of the Deformation Induced Topographic Effect. The value of VGG (Zahorec et al, 2016) can be used to reduce gravity readings to the ground or to a common level in order to compare measurements from various types of gravity meters (relative and / or absolute ones). This procedure is necessary to avoid significant systematic errors. In the same paper, it discusses the significance of the VGG for volcano monitoring. Another example of high precision of VGG determination can be found in (Repanić et al, 2015).

Interesting applications which make use of VGG (Pánisová and Pašteka, 2009) can be found in archaeology, such as the detection of subsurface cavities (for example air filled cavities, water filled cavities), crypts, cellars and tunnels in churches and castles.

In Geodesy (Hackney and Featherstone, 2003) to compute gravity anomaly at the geoid requires the knowledge of the VGG. Gravity anomaly is necessary for the determination of geoid undulation using Stokes' formula.

The vertical gradient of normal gravity (VGNG) - or "free – air" correction - is used to partly downward or upward continue observed gravity to the geoid. As a linear approximation the value of 0, 3086 mgal/m (at geodetic latitude $\varphi = 45^0$ on the surface of the ellipsoid of revolution) is used. A second order approximation takes into account the oblate elliptical shape of the Earth. In this work the expression of the VGNG in geodetic coordinates at a point P on the Earth's physical surface will be presented. This effort will shed some light to the problem of finding a general expression for VGNG in geodetic coordinates. The variation of VGNG along the vertical line to the ellipsoid passing through point P depends also on the variation of the mean curvature of the normal equipotential surfaces. For this reason an effort has been made in order to express the fundamental quantities $E^U$, $F^U$, $G^U$, $L^U$, $M^U$ and $N^U$ of the normal equipotential surfaces as combinations of the fundamental quantities of the ellipsoid of revolution (in the sequel it will be referred as "ellipsoid") .

In section 2 the quantities of the first fundamental form of a normal equipotential surface $U = U_P$ will be determined (at point P), and in section 3 the quantities of the second fundamental form will be determined. Finally, in section 4 the results will be tabulated and some conclusions will be discussed in section 5.

## II. Methodology, quantities of the first fundamental form

Let a point P on the Earth's physical surface with geodetic coordinates ($\varphi_P$, $\lambda_P$, $h_P$). In addition let

$$\bar{s}^e : (\phi_P - \delta\phi, \phi_P + \delta\phi) \times (\lambda_P - \delta\lambda, \lambda_P + \delta\lambda) \to \Re^3 : (\phi, \lambda) \to \bar{s}^e(\phi, \lambda)$$

$$\bar{s}^e(\phi, \lambda) = \left( \frac{a^2}{b\sqrt{1 + e'^2 \cos^2\phi}} \cos\phi \cos\lambda, \frac{a^2}{b\sqrt{1 + e'^2 \cos^2\phi}} \cos\phi \sin\lambda, \frac{b}{\sqrt{1 + e'^2 \cos^2\phi}} \sin\phi \right) \tag{2.1}$$

where

$$e'^2 = \frac{a^2 - b^2}{b^2} \tag{2.2}$$

be a parametric representation of a part of an ellipsoid of revolution. The geometry of the ellipsoid of revolution is studied in detail in (Deakin and Hunter, 2003), therefore the quantities which will be needed in the sequel are

$$\bar{s}^e_\phi(\phi, \lambda) \equiv \bar{s}^e_\phi = \left( -\frac{a^2 \sin\phi \cos\lambda}{b(1 + e'^2 \cos^2\phi)^{\frac{3}{2}}}, -\frac{a^2 \sin\phi \sin\lambda}{b(1 + e'^2 \cos^2\phi)^{\frac{3}{2}}}, \frac{a^2 \cos\phi}{b(1 + e'^2 \cos^2\phi)^{\frac{3}{2}}} \right) \tag{2.3}$$

$$\bar{s}^e_\lambda(\phi, \lambda) \equiv \bar{s}^e_\lambda = \left( -\frac{a^2 \cos\phi \sin\lambda}{b\sqrt{1 + e'^2 \cos^2\phi}}, \frac{a^2 \cos\phi \cos\lambda}{b\sqrt{1 + e'^2 \cos^2\phi}}, 0 \right) \tag{2.4}$$

The unit normal vector and its derivatives are

$$\bar{N}^e(\phi, \lambda) \equiv \bar{N}^e = (-\cos\phi \cos\lambda, -\cos\phi \sin\lambda, -\sin\phi) \tag{2.5}$$

$$\bar{N}^e_\phi(\phi, \lambda) \equiv \bar{N}^e_\phi = (\sin\phi \cos\lambda, \sin\phi \sin\lambda, -\cos\phi) \tag{2.6}$$

$$\bar{N}^e_\lambda(\phi, \lambda) \equiv \bar{N}^e_\lambda = (\cos\phi \sin\lambda, -\cos\phi \cos\lambda, 0) \tag{2.7}$$

$$\bar{N}^e_{\phi\phi}(\phi, \lambda) \equiv \bar{N}^e_{\phi\phi} = (\cos\phi \cos\lambda, \cos\phi \sin\lambda, \sin\phi) = -\bar{N}^e \tag{2.8}$$

$$\bar{N}^e_{\phi\lambda}(\phi, \lambda) \equiv \bar{N}^e_{\phi\lambda} = (-\sin\phi \sin\lambda, \sin\phi \cos\lambda, 0) = -\bar{N}^e_\lambda \tan\phi \tag{2.9}$$

$$\bar{N}^e_{\lambda\lambda}(\phi, \lambda) \equiv \bar{N}^e_{\lambda\lambda} = (\cos\phi \cos\lambda, \cos\phi \sin\lambda, 0) \tag{2.10}$$

The fundamental quantities of the ellipsoid are

$$E^e(\phi) \equiv E^e = \frac{a^4}{b^2(1 + e'^2 \cos^2\phi)^3} \tag{2.11}$$

$$F^e(\phi) \equiv F^e = 0 \tag{2.12}$$

$$G^e(\phi) \equiv G^e = \frac{a^4 \cos^2 \phi}{b^2 (1 + e'^2 \cos^2 \phi)} \tag{2.13}$$

$$L^e(\phi) \equiv L^e = \frac{a^2}{b(1 + e'^2 \cos^2 \phi)^{\frac{3}{2}}} \tag{2.14}$$

$$M^e(\phi) \equiv M^e = 0 \tag{2.15}$$

$$N^e(\phi) \equiv N^e = \frac{a^2 \cos^2 \phi}{b\sqrt{1 + e'^2 \cos^2 \phi}} \tag{2.16}$$

From Weingarten equations (Weatherburn, 1995) we have that

$$\overline{N}^e_\phi = -\frac{L^e}{E^e} \overline{s}^e_\phi \tag{2.17}$$

$$\overline{N}^e_\lambda = -\frac{N^e}{G^e} \overline{s}^e_\lambda \tag{2.18}$$

Known relations hold for the mean and Gauss curvature of the ellipsoid

$$J^e(\phi) \equiv J^e = \frac{1}{2}\left(\frac{L^e}{E^e} + \frac{N^e}{G^e}\right) \tag{2.19}$$

$$K^e_G(\phi) \equiv K^e_G = \frac{L^e N^e}{E^e G^e} \tag{2.20}$$

Now let $S^p$ be a parallel surface of the ellipsoid such that

$$\begin{aligned} &\overline{s}^p : (\phi_P - \delta\phi, \phi_P + \delta\phi) \times (\lambda_P - \delta\lambda, \lambda_P + \delta\lambda) \to \Re^3 : (\phi, \lambda) \to \overline{s}^p(\phi, \lambda) \\ &\overline{s}^p(\phi, \lambda) = \overline{s}^e(\phi, \lambda) - h_P \overline{N}^e(\phi, \lambda) \end{aligned} \tag{2.21}$$

It holds that

$$\overline{s}^p_\phi(\phi, \lambda) \equiv \overline{s}^p_\phi = \overline{s}^e_\phi - h_P \overline{N}^e_\phi \tag{2.22}$$

$$\overline{s}^p_\lambda(\phi, \lambda) \equiv \overline{s}^p_\lambda = \overline{s}^e_\lambda - h_P \overline{N}^e_\lambda \tag{2.23}$$

$$\overline{s}^p_{\phi\phi}(\phi, \lambda) \equiv \overline{s}^p_{\phi\phi} = \overline{s}^e_{\phi\phi} - h_P \overline{N}^e_{\phi\phi} \tag{2.24}$$

$$\overline{s}^p_{\phi\lambda}(\phi, \lambda) \equiv \overline{s}^p_{\phi\lambda} = \overline{s}^e_{\phi\lambda} - h_P \overline{N}^e_{\phi\lambda} \tag{2.25}$$

$$\overline{s}^p_{\lambda\lambda}(\phi, \lambda) \equiv \overline{s}^p_{\lambda\lambda} = \overline{s}^e_{\lambda\lambda} - h_P \overline{N}^e_{\lambda\lambda} \tag{2.26}$$

The unit normal vector of the parallel surface passing through point P is

$$\overline{N}^{p}(\phi,\lambda) \equiv \overline{N}^{p} = \overline{N}^{e} \tag{2.27}$$

The fundamental quantities of the parallel surface are (Kiziltuğ and Tarakci, 2013)

$$E^{p}(\phi) \equiv E^{p} = (1 - h_P^2 K_G^e)E^e + 2h_P(h_P J^e + 1)L^e \tag{2.28}$$

$$F^{p}(\phi) \equiv F^{p} = (1 - h_P^2 K_G^e)F^e + 2h_P(h_P J^e + 1)M^e = 0 \tag{2.29}$$

$$G^{p}(\phi) \equiv G^{p} = (1 - h_P^2 K_G^e)G^e + 2h_P(h_P J^e + 1)N^e \tag{2.30}$$

$$L^{p}(\phi) \equiv L^{e} = (1 + 2h_P J^e)L^e - h_P K_G^e E^e \tag{2.31}$$

$$M^{p}(\phi) \equiv M^{p} = (1 + 2h_P J^e)M^e - h_P K_G^e F^e = 0 \tag{2.32}$$

$$N^{p}(\phi) \equiv N^{p} = (1 + 2h_P J^e)N^e - h_P K_G^e G^e \tag{2.33}$$

The differences in certain signs are due to the fact that the unit normal vector of the ellipsoid points inwards. Therefore the vector equation of the parallel surface is written with a minus sign instead of a plus sign.
Now let $S^U$ be the normal equipotential surface passing through point P. The value of the normal potential on this surface is $U_P$. Let a parameterization of a part of the equipotential surface be

$$\begin{gathered}\overline{s}^{U} : (\phi_P - \delta\phi, \phi_P + \delta\phi) \times (\lambda_P - \delta\lambda, \lambda_P + \delta\lambda) \to \Re^3 : (\phi,\lambda) \to \overline{s}^{U}(\phi,\lambda) \\ \overline{s}^{U}(\phi,\lambda) = \overline{s}^{p}(\phi,\lambda) - \delta h(\phi)\overline{N}^{p}(\phi,\lambda) = \overline{s}^{P} - \delta h \overline{N}^{P}\end{gathered} \tag{2.34}$$

where

$$\delta h : (\phi_P - \delta\phi, \phi_P + \delta\phi) \to \Re : \phi \to \delta h(\phi) : \delta h(\phi_P) = 0 \tag{2.35}$$

It holds that

$$\overline{s}_{\phi}^{U} = \overline{s}_{\phi}^{p} - \delta h_{\phi}\overline{N}^{p} - \delta h \overline{N}_{\phi}^{p} = \overline{s}_{\phi}^{p} - \delta h_{\phi}\overline{N}^{e} - \delta h \overline{N}_{\phi}^{e} \tag{2.36}$$

$$\overline{s}_{\lambda}^{U} = \overline{s}_{\lambda}^{p} - \delta h \overline{N}_{\lambda}^{p} = \overline{s}_{\lambda}^{p} - \delta h \overline{N}_{\lambda}^{e} \tag{2.37}$$

The quantities of the first fundamental form are

$$\begin{aligned}E^{U}(\phi) \equiv E^{U} &= \langle \overline{s}_{\phi}^{U}, \overline{s}_{\phi}^{U} \rangle = \langle \overline{s}_{\phi}^{p} - \delta h_{\phi}\overline{N}^{e} - \delta h \overline{N}_{\phi}^{e}, \overline{s}_{\phi}^{p} - \delta h_{\phi}\overline{N}^{e} - \delta h \overline{N}_{\phi}^{e} \rangle = \\ &= \langle \overline{s}_{\phi}^{p}, \overline{s}_{\phi}^{p} \rangle - 2\delta h \langle \overline{s}_{\phi}^{p}, \overline{N}_{\phi}^{e} \rangle + \delta h_{\phi}^{2} + \delta h^{2} \langle \overline{N}_{\phi}^{e}, \overline{N}_{\phi}^{e} \rangle = \\ &= \langle \overline{s}_{\phi}^{p}, \overline{s}_{\phi}^{p} \rangle + 2\delta h \frac{L^e}{E^e} \langle \overline{s}_{\phi}^{p}, \overline{s}_{\phi}^{e} \rangle + \delta h^2 \left(\frac{L^e}{E^e}\right)^2 \langle \overline{s}_{\phi}^{e}, \overline{s}_{\phi}^{e} \rangle + \delta h_{\phi}^{2} = \\ &= \langle \overline{s}_{\phi}^{p}, \overline{s}_{\phi}^{p} \rangle + 2\delta h \frac{L^e}{E^e} \langle \overline{s}_{\phi}^{e} - h_P \overline{N}_{\phi}^{e}, \overline{s}_{\phi}^{e} \rangle + \delta h^2 \left(\frac{L^e}{E^e}\right)^2 \langle \overline{s}_{\phi}^{e}, \overline{s}_{\phi}^{e} \rangle + \delta h_{\phi}^{2} = \\ &= \langle \overline{s}_{\phi}^{p}, \overline{s}_{\phi}^{p} \rangle + 2\delta h \frac{L^e}{E^e} \langle \overline{s}_{\phi}^{e}, \overline{s}_{\phi}^{e} \rangle + 2h_P \delta h \left(\frac{L^e}{E^e}\right)^2 \langle \overline{s}_{\phi}^{e}, \overline{s}_{\phi}^{e} \rangle + \delta h^2 \left(\frac{L^e}{E^e}\right)^2 \langle \overline{s}_{\phi}^{e}, \overline{s}_{\phi}^{e} \rangle + \delta h_{\phi}^{2}\end{aligned} \tag{2.38}$$

Eventually

$$E^{U} = \left[(1 - h_P^2 K_G^e) E^e + 2\delta h L^e + (2h_P \delta h + \delta h^2)\left(\frac{L^e}{\sqrt{E^e}}\right)^2\right] + 2h_P (h_P J^e + 1) L^e + \delta h_\phi^2 \tag{2.39}$$

Since normal equipotential surfaces are surfaces of revolution then

$$F^{U}(\phi) \equiv F^{U} = 0 \tag{2.40}$$

$$\begin{aligned} G^{U}(\phi) \equiv G^{U} &= \langle \bar{s}_\lambda^p - \delta h \overline{N}_\lambda^e, \bar{s}_\lambda^p - \delta h \overline{N}_\lambda^e \rangle = \langle \bar{s}_\lambda^e - (h_P + \delta h)\overline{N}_\lambda^e, \bar{s}_\lambda^e - (h_P + \delta h)\overline{N}_\lambda^e \rangle = \\ &= \langle \bar{s}_\lambda^e, \bar{s}_\lambda^e \rangle + 2(h_P + \delta h)\frac{N^e}{G^e}\langle \bar{s}_\lambda^e, \bar{s}_\lambda^e \rangle + (h_P + \delta h)^2 \left(\frac{N^e}{G^e}\right)^2 \langle \bar{s}_\lambda^e, \bar{s}_\lambda^e \rangle \end{aligned} \tag{2.41}$$

Eventually

$$G^{U} = G^e + 2(h_P + \delta h) N^e + (h_P + \delta h)^2 \left(\frac{N^e}{\sqrt{G^e}}\right)^2 \tag{2.42}$$

A simple expression for the unit normal vector of the normal equipotential surface is

$$\overline{N}^{U}(\phi) \equiv \overline{N}^{U} = \frac{\bar{s}_\phi^p}{\left|\bar{s}_\phi^p\right|}\sin\varepsilon + \overline{N}^p \cos\varepsilon = \frac{\bar{s}_\phi^e}{\left|\bar{s}_\phi^e\right|}\sin\varepsilon + \overline{N}^e \cos\varepsilon \tag{2.43}$$

The angle ε (Moritz, 1967) – or normal reduction - can be determined from the following relation

$$\varepsilon(\phi, h) \equiv \varepsilon = -\frac{(\gamma_b - \gamma_a) h}{\gamma_a R}\sin 2\phi \quad , \quad R = 6371 \ km \tag{2.44}$$

The symbols $\gamma_\alpha$ and $\gamma_b$ stand for the value of normal gravity at the equator and poles respectively. The minus sign is conventional and it is adopted in eq. (2.43). One issue in eq. (2.39) is that the function $\delta h_\varphi$ is unknown, therefore it must be determined. It holds that

$$\begin{aligned} \bar{s}_\phi^U \times \bar{s}_\lambda^U &= (\bar{s}_\phi^p - \delta h_\phi \overline{N}^e - \delta h \overline{N}_\phi^e) \times (\bar{s}_\lambda^p - \delta h \overline{N}_\lambda^e) = \\ &= (\bar{s}_\phi^p \times \bar{s}_\lambda^p) - \delta h (\bar{s}_\phi^p \times \overline{N}_\lambda^e) - \delta h_\phi (\overline{N}^e \times \bar{s}_\lambda^p) + \delta h_\phi \delta h (\overline{N}^e \times \overline{N}_\lambda^e) - \\ &\quad - \delta h (\overline{N}_\phi^e \times \bar{s}_\lambda^p) + \delta h^2 (\overline{N}_\phi^e \times \overline{N}_\lambda^e) \end{aligned} \tag{2.45}$$

But

$$\begin{aligned} \bar{s}_\phi^p \times \bar{s}_\lambda^p &= (\bar{s}_\phi^e - h_P \overline{N}_\phi^e) \times (\bar{s}_\lambda^e - h_P \overline{N}_\lambda^e) = (\bar{s}_\phi^e \times \bar{s}_\lambda^e) + h_P \frac{N^e}{G^e}(\bar{s}_\phi^e \times \bar{s}_\lambda^e) + h_P \frac{L^e}{E^e}(\bar{s}_\phi^e \times \bar{s}_\lambda^e) + \\ &\quad + h_P^2 \frac{L^e N^e}{E^e G^e}(\bar{s}_\phi^e \times \bar{s}_\lambda^e) = (1 + 2h_P J^e + h_P^2 K_G^e)(\bar{s}_\phi^e \times \bar{s}_\lambda^e) \end{aligned} \tag{2.46}$$

$$\bar{s}^{p}_{\phi} \times \bar{N}^{e}_{\lambda} = \left(\bar{s}^{e}_{\phi} + h_P \frac{L^e}{E^e}\bar{s}^{e}_{\phi}\right) \times \left(-\frac{N^e}{G^e}\bar{s}^{e}_{\lambda}\right) = -\left(h_P K^e_G + \frac{N^e}{G^e}\right)(\bar{s}^{e}_{\phi} \times \bar{s}^{e}_{\lambda}) \tag{2.47}$$

$$\begin{aligned} \bar{N}^{e} \times \bar{s}^{p}_{\lambda} &= \frac{1}{\sqrt{E^e G^e}}(\bar{s}^{e}_{\phi} \times \bar{s}^{e}_{\lambda}) \times (\bar{s}^{e}_{\lambda} - h_P \bar{N}^{e}_{\lambda}) = \frac{1}{\sqrt{E^e G^e}}(\bar{s}^{e}_{\phi} \times \bar{s}^{e}_{\lambda}) \times \left(1 + h_P \frac{N^e}{G^e}\right)\bar{s}^{e}_{\lambda} = \\ &= \frac{1}{\sqrt{E^e G^e}}\left(1 + h_P \frac{N^e}{G^e}\right)[(\bar{s}^{e}_{\phi} \times \bar{s}^{e}_{\lambda}) \times \bar{s}^{e}_{\lambda}] = -\frac{\sqrt{G^e}}{\sqrt{E^e}}\left(1 + h_P \frac{N^e}{G^e}\right)\bar{s}^{e}_{\phi} \end{aligned} \tag{2.48}$$

$$\bar{N}^{e} \times \bar{N}^{e}_{\lambda} = -\frac{N^e}{G^e\sqrt{E^e G^e}}[(\bar{s}^{e}_{\phi} \times \bar{s}^{e}_{\lambda}) \times \bar{s}^{e}_{\lambda}] = \frac{N^e}{\sqrt{E^e G^e}}\bar{s}^{e}_{\phi} \tag{2.49}$$

$$\begin{aligned} \bar{N}^{e}_{\phi} \times \bar{s}^{p}_{\lambda} &= \left(-\frac{L^e}{E^e}\bar{s}^{e}_{\phi}\right) \times \left(\bar{s}^{e}_{\lambda} + h_P \frac{N^e}{G^e}\bar{s}^{e}_{\lambda}\right) = \left(-\frac{L^e}{E^e}\bar{s}^{e}_{\phi}\right) \times \left[\left(1 + h_P \frac{N^e}{G^e}\right)\bar{s}^{e}_{\lambda}\right] = \\ &= -\left(\frac{L^e}{E^e} + h_P K^e_G\right)(\bar{s}^{e}_{\phi} \times \bar{s}^{e}_{\lambda}) \end{aligned} \tag{2.50}$$

$$\bar{N}^{e}_{\phi} \times \bar{N}^{e}_{\lambda} = \frac{L^e N^e}{E^e G^e}(\bar{s}^{e}_{\phi} \times \bar{s}^{e}_{\lambda}) = K^e_G(\bar{s}^{e}_{\phi} \times \bar{s}^{e}_{\lambda}) \tag{2.51}$$

Hence relation (2.45) becomes

$$\begin{aligned} \bar{s}^{U}_{\phi} \times \bar{s}^{U}_{\lambda} &= (1 + 2h_P J^e + h_P^2 K^e_G)(\bar{s}^{e}_{\phi} \times \bar{s}^{e}_{\lambda}) + \delta h\left(h_P K^e_G + \frac{N^e}{G^e}\right)(\bar{s}^{e}_{\phi} \times \bar{s}^{e}_{\lambda}) + \\ &+ \delta h_\phi \frac{\sqrt{G^e}}{\sqrt{E^e}}\left(1 + h_P \frac{N^e}{G^e}\right)\bar{s}^{e}_{\phi} + \delta h_\phi \delta h \frac{N^e}{\sqrt{E^e G^e}}\bar{s}^{e}_{\phi} + \delta h\left(\frac{L^e}{E^e} + h_P K^e_G\right)(\bar{s}^{e}_{\phi} \times \bar{s}^{e}_{\lambda}) + \\ &+ \delta h^2 K^e_G(\bar{s}^{e}_{\phi} \times \bar{s}^{e}_{\lambda}) \end{aligned} \tag{2.52}$$

Rearranging terms, the components of the above vector are equal to

$$\begin{aligned} \bar{s}^{U}_{\phi} \times \bar{s}^{U}_{\lambda} &= \sqrt{E^e G^e}[(1 + 2h_P J^e + h_P^2 K^e_G) + 2\delta h(h_P K^e_G + J^e) + \delta h^2 K^e_G]\bar{N}^{e} + \\ &+ \delta h_\phi\left[\sqrt{G^e}\left(1 + h_P \frac{N^e}{G^e}\right) + \delta h \frac{N^e}{\sqrt{G^e}}\right]\frac{\bar{s}^{e}_{\phi}}{\left|\bar{s}^{e}_{\phi}\right|} \end{aligned} \tag{2.53}$$

Therefore (see eq. (2.13) and (2.16))

$$\tan\varepsilon = \frac{\delta h_\phi\left[\sqrt{G^e}\left(1 + h_P \frac{N^e}{G^e}\right) + \delta h\cos\phi\right]}{\sqrt{E^e G^e}[(1 + 2h_P J^e + h_P^2 K^e_G) + 2\delta h(h_P K^e_G + J^e) + \delta h^2 K^e_G]} \tag{2.54}$$

Thus

$$\delta h_{\phi} = \frac{\sqrt{E^{e}G^{e}}[(1+2h_{P}J^{e}+h_{P}^{2}K_{G}^{e})+2\delta h(h_{P}K_{G}^{e}+J^{e})+\delta h^{2}K_{G}^{e}]\tan\varepsilon}{\left[\sqrt{G^{e}}\left(1+h_{P}\frac{N^{e}}{G^{e}}\right)+\delta h\cos\phi\right]} \tag{2.55}$$

At point P a simpler formula holds since $\delta h(\varphi_P) = 0$ hence

$$\delta h_{\phi}(\phi_{P}) = \left.\frac{\sqrt{E^{e}}(1+2h_{P}J^{e}+h_{P}^{2}K_{G}^{e})\tan\varepsilon}{\left(1+h_{P}\frac{N^{e}}{G^{e}}\right)}\right|_{\phi_P} \tag{2.56}$$

In eq. (2.55) and (2.56) the minus sign to the angle ε (see eq. (2.44)) is adopted.

## III. Methodology, quantities of the second fundamental form

For the quantities $L^U$, $M^U$ and $N^U$ the following second order partial derivatives are needed (see eq. (2.8),(2.9), (2.36), and (2.37))

$$\begin{aligned}\bar{s}_{\phi\phi}^{U} &= \bar{s}_{\phi\phi}^{p} - \delta h_{\phi\phi}\overline{N}^{p} - 2\delta h_{\phi}\overline{N}_{\phi}^{p} - \delta h\overline{N}_{\phi\phi}^{p} = \bar{s}_{\phi\phi}^{e} - h_{P}\overline{N}_{\phi\phi}^{e} - \delta h_{\phi\phi}\overline{N}^{e} - 2\delta h_{\phi}\overline{N}_{\phi}^{e} - \delta h\overline{N}_{\phi\phi}^{e} = \\ &= \bar{s}_{\phi\phi}^{e} + (h_{P}+\delta h-\delta h_{\phi\phi})\overline{N}^{e} + 2\delta h_{\phi}\frac{L^{e}}{E^{e}}\bar{s}_{\phi}^{e}\end{aligned} \tag{3.1}$$

$$\begin{aligned}\bar{s}_{\phi\lambda}^{U} &= \bar{s}_{\phi\lambda}^{p} - \delta h_{\phi}\overline{N}_{\lambda}^{p} - \delta h\overline{N}_{\phi\lambda}^{p} = \bar{s}_{\phi\lambda}^{p} - \delta h_{\phi}\overline{N}_{\lambda}^{e} - \delta h\overline{N}_{\phi\lambda}^{e} = \bar{s}_{\phi\lambda}^{p} - \delta h_{\phi}\overline{N}_{\lambda}^{e} + \delta h\overline{N}_{\lambda}^{e}\tan\phi = \\ &= \bar{s}_{\phi\lambda}^{e} - h_{P}\overline{N}_{\phi\lambda}^{e} - \delta h_{\phi}\overline{N}_{\lambda}^{e} + \delta h\overline{N}_{\lambda}^{e}\tan\phi = \bar{s}_{\phi\lambda}^{e} + h_{P}\overline{N}_{\lambda}^{e}\tan\phi - \delta h_{\phi}\overline{N}_{\lambda}^{e} + \delta h\overline{N}_{\lambda}^{e}\tan\phi = \\ &= \bar{s}_{\phi\lambda}^{e} - \frac{N^{e}}{G^{e}}[(h_{P}+\delta h)\tan\phi - \delta h_{\phi}]\bar{s}_{\lambda}^{e}\end{aligned} \tag{3.2}$$

$$\bar{s}_{\lambda\lambda}^{U} = \bar{s}_{\lambda\lambda}^{p} - \delta h\overline{N}_{\lambda\lambda}^{p} = \bar{s}_{\lambda\lambda}^{e} - h_{P}\overline{N}_{\lambda\lambda}^{e} - \delta h\overline{N}_{\lambda\lambda}^{e} = \bar{s}_{\lambda\lambda}^{e} - (h_{P}+\delta h)\overline{N}_{\lambda\lambda}^{e} \tag{3.3}$$

Hence

$$\begin{aligned}L^{U}(\phi) &\equiv L^{U} = \langle\overline{N}^{U},\bar{s}_{\phi\phi}^{U}\rangle = \langle\frac{\bar{s}_{\phi}^{e}}{\left|\bar{s}_{\phi}^{e}\right|}\sin\varepsilon + \overline{N}^{e}\cos\varepsilon, \bar{s}_{\phi\phi}^{e} + (h_{P}+\delta h-\delta h_{\phi\phi})\overline{N}^{e} + 2\delta h_{\phi}\frac{L^{e}}{E^{e}}\bar{s}_{\phi}^{e}\rangle = \\ &= \frac{\sin\varepsilon}{\left|\bar{s}_{\phi}^{e}\right|}\langle\bar{s}_{\phi}^{e},\bar{s}_{\phi\phi}^{e}\rangle + \frac{2\delta h_{\phi}\sin\varepsilon}{\left|\bar{s}_{\phi}^{e}\right|}\frac{L^{e}}{E^{e}}\langle\bar{s}_{\phi}^{e},\bar{s}_{\phi}^{e}\rangle + \langle\overline{N}^{e},\bar{s}_{\phi\phi}^{e}\rangle\cos\varepsilon + (h_{P}+\delta h-\delta h_{\phi\phi})\cos\varepsilon\end{aligned} \tag{3.4}$$

The first inner product is equal to

$$\langle\bar{s}_{\phi}^{e},\bar{s}_{\phi\phi}^{e}\rangle = \frac{1}{2}(\langle\bar{s}_{\phi}^{e},\bar{s}_{\phi}^{e}\rangle)_{\phi} = \frac{1}{2}E_{\phi}^{e} \tag{3.5}$$

Therefore

$$L^{U} = \frac{E_{\phi}^{e} + 4\delta h_{\phi}L^{e}}{2\sqrt{E^{e}}}\sin\varepsilon + (L^{e} + h_{P} + \delta h - \delta h_{\phi\phi})\cos\varepsilon \tag{3.6}$$

The function $\delta h_{\varphi\varphi}$ can be found from the derivation of eq. (2.55). The following two derivatives are needed

$$(\sqrt{E^eG^e})_\phi = \frac{E^e_\phi G^e + E^e G^e_\phi}{2\sqrt{E^eG^e}} \tag{3.7}$$

$$\left[\sqrt{G^e}\left(1 + h_P\frac{N^e}{G^e}\right) + \delta h \cos\phi\right]_\phi = \frac{G^e_\phi}{2\sqrt{G^e}}\left(1 + h_P\frac{N^e}{G^e}\right) + h_P\frac{N^e_\phi G^e - N^e G^e_\phi}{(G^e)^{\frac{3}{2}}} + \\ + \delta h_\phi \cos\phi - \delta h \sin\phi \tag{3.8}$$

The derivative of eq. (2.55) is

$$\delta h_{\phi\phi} = \frac{(E^e_\phi G^e + E^e G^e_\phi)[(1 + 2h_P J^e + h_P^2 K^e_G) + 2\delta h(h_P K^e_G + J^e) + \delta h^2 K^e_G]\tan\varepsilon}{2\sqrt{E^eG^e}\left[\sqrt{G^e}\left(1 + h_P\frac{N^e}{G^e}\right) + \delta h\cos\phi\right]} + \\
+ \frac{\sqrt{E^eG^e}[(1 + 2h_P J^e + h_P^2 K^e_G) + 2\delta h(h_P K^e_G + J^e) + \delta h^2 K^e_G]_\phi \tan\varepsilon}{\left[\sqrt{G^e}\left(1 + h_P\frac{N^e}{G^e}\right) + \delta h\cos\phi\right]} + \\
+ \frac{\sqrt{E^eG^e}[(1 + 2h_P J^e + h_P^2 K^e_G) + 2\delta h(h_P K^e_G + J^e) + \delta h^2 K^e_G]\varepsilon_\phi}{\left[\sqrt{G^e}\left(1 + h_P\frac{N^e}{G^e}\right) + \delta h\cos\phi\right]\cos^2\varepsilon} - \\
- \{\sqrt{E^eG^e}[(1 + 2h_P J^e + h_P^2 K^e_G) + 2\delta h(h_P K^e_G + J^e) + \delta h^2 K^e_G]\tan\varepsilon\}\cdot \\
\left[\frac{G^e_\phi}{2\sqrt{G^e}}\left(1 + h_P\frac{N^e}{G^e}\right) + h_P\frac{N^e_\phi G^e - N^e G^e_\phi}{(G^e)^{\frac{3}{2}}} + \delta h_\phi\cos\phi - \delta h\sin\phi\right]\cdot \\
\left[\sqrt{G^e}\left(1 + h_P\frac{N^e}{G^e}\right) + \delta h\cos\phi\right]^{-2} \tag{3.9}$$

At point P δh = 0 hence

$$\delta h_{\phi\phi}(\phi_P) = \left\{\frac{(E^e_\phi G^e + E^e G^e_\phi)\tan\varepsilon\cos^2\varepsilon + 2E^eG^e\varepsilon_\phi}{2\sqrt{E^eG^e}\left[\sqrt{G^e}\left(1 + h_P\frac{N^e}{G^e}\right)\right]\cos^2\varepsilon}\right\}_{\phi_P}(1 + 2h_P J^e + h_P^2 K^e_G)_{\phi_P} - \\
- \left.\frac{\sqrt{E^eG^e}\left[\frac{G^e_\phi}{2\sqrt{G^e}}\left(1 + h_P\frac{N^e}{G^e}\right) + h_P\frac{N^e_\phi G^e - N^e G^e_\phi}{(G^e)^{\frac{3}{2}}} + \delta h_\phi\cos\phi\right](1 + 2h_P J^e + h_P^2 K^e_G)\tan\varepsilon}{G^e\left(1 + h_P\frac{N^e}{G^e}\right)^2}\right|_{\phi_P} + \\
+ \left.\frac{\sqrt{E^eG^e}[(1 + 2h_P J^e_\phi + h_P^2 K^e_{G,\phi}) + 2\delta h_\phi(h_P K^e_G + J^e)]\tan\varepsilon}{\sqrt{G^e}\left(1 + h_P\frac{N^e}{G^e}\right)}\right|_{\phi_P} \tag{3.9a}$$

Therefore at point P (setting δh = 0 to eq. (3.6))

$$L^U(\phi_P) = \left.\frac{E^e_\phi + 4\delta h_\phi L^e}{2\sqrt{E^e}}\right|_{\phi_P} \sin\varepsilon_P + (L^e(\phi_P) + h_P)\cos\varepsilon_P - \delta h_{\phi\phi}(\phi_P)\cos\varepsilon_P \tag{3.10}$$

$$\begin{aligned} M^U(\phi) \equiv M^U &= \langle \frac{\bar{s}^e_\phi}{\left|\bar{s}^e_\phi\right|}\sin\varepsilon + \bar{N}^e\cos\varepsilon, \bar{s}^e_{\phi\lambda} - \frac{N^e}{G^e}[(h_P + \delta h)\tan\phi - \delta h_\phi]\bar{s}^e_\lambda\rangle = \\ &= \frac{\sin\varepsilon}{\left|\bar{s}^e_\phi\right|}\langle \bar{s}^e_\phi, \bar{s}^e_{\phi\lambda}\rangle + 0 + M^e\cos\varepsilon + 0 \end{aligned} \tag{3.11}$$

But

$$\bar{s}^e_{\phi\lambda} = \left(\frac{a^2\sin\phi\sin\lambda}{b(1+e'^2\cos^2\phi)^{\frac{3}{2}}}, -\frac{a^2\sin\phi\cos\lambda}{b(1+e'^2\cos^2\phi)^{\frac{3}{2}}}, 0\right) \tag{3.12}$$

and

$$\begin{aligned} \langle \bar{s}^e_\phi, \bar{s}^e_{\phi\lambda}\rangle = \langle &\left(-\frac{a^2\sin\phi\cos\lambda}{b(1+e'^2\cos^2\phi)^{\frac{3}{2}}}, -\frac{a^2\sin\phi\sin\lambda}{b(1+e'^2\cos^2\phi)^{\frac{3}{2}}}, \frac{a^2\cos\phi}{b(1+e'^2\cos^2\phi)^{\frac{3}{2}}}\right), \\ &,\left(\frac{a^2\sin\phi\sin\lambda}{b(1+e'^2\cos^2\phi)^{\frac{3}{2}}}, -\frac{a^2\sin\phi\cos\lambda}{b(1+e'^2\cos^2\phi)^{\frac{3}{2}}}, 0\right)\rangle = 0 \end{aligned} \tag{3.13}$$

From the above relation and eq. (2.15) we have that

$$M^U = 0 \tag{3.14}$$

$$\begin{aligned} N^U(\phi) \equiv N^U &= \langle \frac{\bar{s}^e_\phi}{\left|\bar{s}^e_\phi\right|}\sin\varepsilon + \bar{N}^e\cos\varepsilon, \bar{s}^e_{\lambda\lambda} - (h_P + \delta h)\bar{N}^e_{\lambda\lambda}\rangle = \\ &= \frac{\sin\varepsilon}{\left|\bar{s}^e_\phi\right|}\langle \bar{s}^e_\phi, \bar{s}^e_{\lambda\lambda}\rangle - (h_P + \delta h)\frac{\sin\varepsilon}{\left|\bar{s}^e_\phi\right|}\langle \bar{s}^e_\phi, \bar{N}^e_{\lambda\lambda}\rangle + \langle \bar{N}^e, \bar{s}^e_{\lambda\lambda}\rangle\cos\varepsilon - (h_P + \delta h)\langle \bar{N}^e, \bar{N}^e_{\lambda\lambda}\rangle\cos\varepsilon \end{aligned} \tag{3.15}$$

$$\begin{aligned} \langle \bar{s}^e_\phi, \bar{s}^e_{\lambda\lambda}\rangle = \langle &\left(-\frac{a^2\sin\phi\cos\lambda}{b(1+e'^2\cos^2\phi)^{\frac{3}{2}}}, -\frac{a^2\sin\phi\sin\lambda}{b(1+e'^2\cos^2\phi)^{\frac{3}{2}}}, \frac{a^2\cos\phi}{b(1+e'^2\cos^2\phi)^{\frac{3}{2}}}\right), \\ &,\left(-\frac{a^2\cos\phi\cos\lambda}{b\sqrt{1+e'^2\cos^2\phi}}, -\frac{a^2\cos\phi\sin\lambda}{b\sqrt{1+e'^2\cos^2\phi}}, 0\right)\rangle = -\frac{a^4\sin\phi\cos\phi}{b^2(1+e'^2\cos^2\phi)^2} \end{aligned} \tag{3.16}$$

$$\left|\bar{s}^e_\phi\right| = \frac{a^2}{b}\sqrt{\frac{1}{(1+e'^2\cos^2\phi)^3}} = \frac{a^2}{b(1+e'^2\cos^2\phi)^{\frac{3}{2}}} \tag{3.17}$$

$$\frac{\langle \bar{s}^e_\phi, \bar{s}^e_{\lambda\lambda}\rangle}{\left|\bar{s}^e_\phi\right|} = -\frac{a^4 \sin\phi\cos\phi}{b^2(1+e'^2\cos^2\phi)^2}\frac{b(1+e'^2\cos^2\phi)^{\frac{3}{2}}}{a^2} = -\frac{a^2\sin\phi\cos\phi}{b\sqrt{(1+e'^2\cos^2\phi)}} = -N^e\tan\phi \tag{3.18}$$

In addition

$$\langle \bar{s}^e_\phi, \bar{N}^e_{\lambda\lambda}\rangle = \langle\left(-\frac{a^2\sin\phi\cos\lambda}{b(1+e'^2\cos^2\phi)^{\frac{3}{2}}}, -\frac{a^2\sin\phi\sin\lambda}{b(1+e'^2\cos^2\phi)^{\frac{3}{2}}}, \frac{a^2\cos\phi}{b(1+e'^2\cos^2\phi)^{\frac{3}{2}}}\right),$$
$$,(\cos\phi\cos\lambda, \cos\phi\sin\lambda, 0)\rangle = -\frac{a^2\sin\phi\cos\phi}{b(1+e'^2\cos^2\phi)^{\frac{3}{2}}} \tag{3.19}$$

$$\frac{\langle \bar{s}^e_\phi, \bar{N}^e_{\lambda\lambda}\rangle}{\left|\bar{s}^e_\phi\right|} = -\frac{a^2\sin\phi\cos\phi}{b(1+e'^2\cos^2\phi)^{\frac{3}{2}}}\frac{b(1+e'^2\cos^2\phi)^{\frac{3}{2}}}{a^2} = -\sin\phi\cos\phi \tag{3.20}$$

$$\langle \bar{N}^e, \bar{N}^e_{\lambda\lambda}\rangle = \langle(-\cos\phi\cos\lambda, -\cos\phi\sin\lambda, -\sin\phi), (\cos\phi\cos\lambda, \cos\phi\sin\lambda, 0)\rangle =$$
$$= -\cos^2\phi \tag{3.21}$$

From eq. (3.16), (3.17), (3.18), (3.19), (3.20) and (3.21) we have eventually that

$$N^U = -N^e\tan\phi\sin\varepsilon + (h_P+\delta h)\sin\phi\cos\phi\sin\varepsilon + N^e\cos\varepsilon + (h_P+\delta h)\cos^2\phi\cos\varepsilon =$$
$$= N^e(\cos\varepsilon - \tan\phi\sin\varepsilon) + (h_P+\delta h)(\sin\phi\cos\phi\sin\varepsilon + \cos^2\phi\cos\varepsilon) \tag{3.22}$$

After some manipulations, at point P the following relation holds

$$N^U(\phi_P) = N^e(\phi_P)(\cos\varepsilon_P - \tan\phi_P\sin\varepsilon_P) + h_P(\sin\phi_P\cos\phi_P\sin\varepsilon_P + \cos^2\phi_P\cos\varepsilon_P) =$$
$$= N^e(\phi_P)(\cos\varepsilon_P - \tan\phi_P\sin\varepsilon_P) + h_P\cos\phi_P\cos(\phi_P - \varepsilon) \tag{3.23}$$

## IV. Results, normal vertical gradient at point P

The following two tables contain all the necessary geometrical quantities for the ellipsoid and the normal equipotential surface U = U$_P$. The quantities which are related to the results of the paper are in Table No 2.

| | | | |
|---|---|---|---|
| $E^e$ | $\frac{a^4}{b^2(1+e'^2\cos^2\phi)^3}$ | $E^e_\varphi$ | $3E^e\frac{e'^2\sin 2\phi}{1+e'^2\cos^2\phi}$ |
| $F^e$ | 0 | $F^e_\varphi$ | 0 |
| $G^e$ | $\frac{a^4\cos^2\phi}{b^2(1+e'^2\cos^2\phi)}$ | $G^e_\varphi$ | $-2G^e\tan\phi + G^e\frac{e'^2\sin 2\phi}{1+e'^2\cos^2\phi}$ |
| $L^e$ | $\frac{a^2}{b(1+e'^2\cos^2\phi)^{\frac{3}{2}}}$ | $L^e_\varphi$ | $\frac{3be'^2}{2a^2}E^e\sqrt{1+e'^2\cos^2\phi}\sin 2\phi$ |
| $M^e$ | 0 | $M^e_\varphi$ | 0 |
| $N^e$ | $\frac{a^2\cos^2\phi}{b\sqrt{1+e'^2\cos^2\phi}}$ | $N^e_\varphi$ | $-2N^e\tan\phi + N^e\frac{e'^2\sin 2\phi}{2(1+e'^2\cos^2\phi)}$ |

**Table No 1:** Fundamental quantities of the ellipsoid and their first derivatives

The quantities in Table No 1 are necessary for the expression of the quantities in Table No 2.

| | |
|---|---|
| $E^U(\varphi_P)$ | $(1-h_P^2K_G^e)E^e(\phi_P)+2h_P(h_PJ^e(\phi_P)+1)L^e(\phi_P)+\delta h_\phi^2(\phi_P)$ |
| $F^U(\varphi_P)$ | $0$ |
| $G^U(\varphi_P)$ | $G^e(\phi_P)+2h_PN^e(\phi_P)+h_P^2\left(\frac{N^e}{\sqrt{G^e}}\right)^2_{\phi_P}$ |
| $L^U(\varphi_P)$ | $\left.\frac{E_\phi^e+4\delta h_\phi L^e}{2\sqrt{E^e}}\right\vert_{\phi_P}\sin\varepsilon_P+(L^e(\phi_P)+h_P)\cos\varepsilon_P-\delta h_{\phi\phi}(\phi_P)\cos\varepsilon_P$ |
| $M^U(\varphi_P)$ | $0$ |
| $N^U(\varphi_P)$ | $N^e(\phi_P)(\cos\varepsilon_P-\tan\phi_P\sin\varepsilon_P)+h_P\cos\phi_P\cos(\phi_P-\varepsilon)$ |
| $\varepsilon(\varphi_P,h_P)$ | $-\frac{(\gamma_b-\gamma_a)h_P}{\gamma_aR}\sin 2\phi_P \quad , \quad R=6371\ km$ |
| $\delta h_\varphi(\varphi_P)$ | $\left.\frac{\sqrt{E^e}(1+2h_PJ^e+h_P^2K_G^e)\tan\varepsilon}{\left(1+h_P\frac{N^e}{G^e}\right)}\right\vert_{\phi_P}$ |
| $\delta h_{\varphi\varphi}(\varphi_P)$ | $\delta h_{\phi\phi}(\phi_P)=\left\{\frac{(E_\phi^eG^e+E^eG_\phi^e)\tan\varepsilon\cos^2\varepsilon+2E^eG^e\varepsilon_\phi}{2\sqrt{E^eG^e}\left[\sqrt{G^e}\left(1+h_P\frac{N^e}{G^e}\right)\right]\cos^2\varepsilon}\right\}_{\phi_P}(1+2h_PJ^e+h_P^2K_G^e)_{\phi_P}-$ <br> $-\left.\frac{\sqrt{E^eG^e}\left[\frac{G_\phi^e}{2\sqrt{G^e}}\left(1+h_P\frac{N^e}{G^e}\right)+h_P\frac{N_\phi^eG^e-N^eG_\phi^e}{(G^e)^{\frac{3}{2}}}+\delta h_\phi\cos\phi\right](1+2h_PJ^e+h_P^2K_G^e)\tan\varepsilon}{G^e\left(1+h_P\frac{N^e}{G^e}\right)^2}\right\vert_{\phi_P}+$ <br> $+\left.\frac{\sqrt{E^eG^e}[(1+2h_PJ_\phi^e+h_P^2K_{G,\phi}^e)+2\delta h_\phi(h_PK_G^e+J^e)]\tan\varepsilon}{\sqrt{G^e}\left(1+h_P\frac{N^e}{G^e}\right)}\right\vert_{\phi_P}$ |

**Table No 2:** Fundamental quantities of the normal equipotential surface U = $U_P$ at point P.

The fundamental quantities of the normal equipotential surface U = $U_P$ at point P contain the fundamental quantities of the ellipsoid. Let Q be the projection poin of point P along the vertical line to the ellipsoid. From the form of the relations in Table No 2 it is evident that as the geometric height tends to zero the fundamental quantities of the aforementioned surface at point P tend to be equal to the fundamental quantities of the ellipsoid at point Q. The most complicated fundamental quantity is $L^U$ since it contains the quantity $\delta h_{\varphi\varphi}$.
For small elevation $h_P$ up above the ellipsoid the following formula is adopted (Moritz, 1967)

$$\gamma(\phi_P,h_P)=\gamma(\phi_P)-\frac{2\gamma(\phi_P)}{a}(1+f+m-2f^2\sin^2\phi_P)h_P+\frac{3G_cM_E}{a^4}h_P^2 \qquad (4.1)$$

where $G_c$ is the gravitational constant, $M_E$ is the Earth's mass, f is the flattening of the ellipsoid, m is given by

$$m = \frac{\omega^2 a^2 b}{G_c M_E} \tag{4.2}$$

where ω is the Earth's mean angular velocity and

$$\gamma(\phi_P) = \frac{a\gamma_a \cos^2\phi_P + b\gamma_b \sin^2\phi_P}{\sqrt{a^2\cos^2\phi_P + b^2\sin^2\phi_P}} \tag{4.2a}$$

The normal vertical gradient of gravity at point P is equal to

$$\left.\frac{\partial\gamma}{\partial h}\right|_P = -2\omega^2 - 2\gamma(\phi_P, h_P)J^U(\phi_P) \tag{4.3}$$

where $J^U$ is the mean curvature of the normal equipotential surface $U = U_P$ at point P. The above relation as a function of geodetic coordinates is very complicated. Relation (4.3) can be written as

$$\left.\frac{\partial\gamma}{\partial h}\right|_P = -2\omega^2 - 2\gamma(\phi_P, h_P)\left(\frac{L^U G^U + E^U N^U}{2E^U G^U}\right) \tag{4.4}$$

If the axis "a" is increasing and "b" is constant (see Table No 1 and Table No 2) then the effect of the geometry of the ellipsoid on VGNG becomes more pronounced since the denominator in the expression of the mean curvature (eq. (4.4)) depends strongly on the geometry of the ellipsoid (see Table No 1 and Table 2).

## V. Conclusions

The VGNG is an interesting quantity both from a practical and a theoretical point of view. In this theoretical study of VGNG the geometry of the normal equipotential surfaces was investigated. This geometry was split into two parts: The first part was the purely geometric part and was represented by the surface geometry of the ellipsoid i.e. its fundamental quantities, its mean and Gauss curvature. The second part was the physical part which was represented by the normal reduction (angle ε) which is the angle between the gravity vector at a point P (located on the Earth's physical surface) and the vertical line to the ellipsoid passing through the same point.

This investigation has shown that the fundamental quantities, the mean and Gauss curvature of the normal equipotential surfaces are very complicated functions of geodetic coordinates. This results into an equivalent complexity of the VGNG as a function of geodetic coordinates. This is a kind of unfortunate event since this kind of formula is not suitable for calculations.

Finally it was found that the VGNG can be strongly affected by the geometry of the ellipsoid if the axles of the ellipsoid are significantly different in magnitude, i.e. axis "a" is much greater that axis "b".

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